SHARP CONDITIONS FOR NONOSCILLATION OF FUNCTIONAL EQUATIONS [1]

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ABSTRACT

Consider the second order linear functional equation

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \tag{$*$}$$

where $P, Q \in C([0,\infty),[0,\infty)), g \in C([0,\infty),R), g(t)$ is increasing, $g(t) > t$ or $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$, and the linear functional equation

$$x(t) - px(t-\tau) + q(t)x(t-\sigma) = 0, \tag{$**$}$$

where $p, \tau, \sigma \in (0,\infty), q(t) \in C([0,\infty),[0,\infty))$. We establish the following "sharp" nonoscillation criteria for Eq. $(*)$ and Eq. $(**)$:

Theorem 1. *If $Q(t)P(g(t)) \le 1/4$ for large t, then Eq. $(*)$ has a nonoscillatory solution.*

Theorem 2. *If $\sigma > \tau$ and for large t*

$$p^{-\sigma/\tau} \cdot q(t) \le \left(\frac{\sigma-\tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma-\tau}{\tau}\right)^{-1},$$

then Eq. $(**)$ has a nonoscillatory solution.

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1. INTRODUCTION

The oscillatory properties of solutions of differential equations with deviating arguments and difference equations with discrete arguments have been the subject of many recent investigations. See, for example, [1,3-5,7,9-11,13,14,18] and the references cited therein. For the oscillatory properties of solutions of functional equations which include difference equations with continuous arguments, the reader is referred to [2,6,12,15,17,19-22].

In 1992, Ladas, Pakula and Wang [12] considered the difference equation

$$x(t)+p_1x(t-\tau_1)+p_2x(t-\tau_2)=0, \quad p_1,p_2,\tau_1,\tau_2\in R \tag{1.1}$$

and proved that every continuous solution of Eq. (1.1) oscillates if and only if the characteristic equation

$$1+p_1e^{-\lambda\tau_1}+p_2e^{-\lambda\tau_2}=0 \tag{1.2}$$

has no real roots. Observe that when $p_1,p_2\in(0,\infty)$, every solution of Eq. (1.1) oscillates. Without loss of generality, it can be assumed that $\tau_1>\tau_2>0$. But then $p_1>0$ is a necessary condition for all solutions of Eq. (1.1) to oscillate. On the basis of this discussion they studied the equation

$$x(t)-px(t-\tau)+qx(t-\sigma)=0, \tag{1.3}$$

where

$$p,q,\tau,\sigma\in(0,\infty) \quad \text{and} \quad \tau<\sigma,$$

and derived the following necessary and sufficient oscillation condition

$$q^{\tau}\sigma^{\sigma}>p^{\sigma}\tau^{\tau}(\sigma-\tau)^{\sigma-\tau}. \tag{1.4}$$

In 1993 Domshlak [2], in 1995 Zhang and Yan [20], in 1996 Shen [17], in 1997 Zhang, Yan and Zhao [22] and in 1998 Zhang, Yan and Choi [21] studied such equations with variable coefficients, while in 1999, Yan and Zhang [19] considered a system of delay difference equations with constant coefficients. Here, we mention the paper [22] in which the authors considered the difference equation with a variable coefficient of the form

$$x(t)-x(t-\tau)+q(t)x(t-\sigma)=0, \tag{1.5}$$

where

$$\tau,\sigma\in(0,\infty),\tau<\sigma \quad \text{and} \quad q(t)\in C([0,\infty),(0,\infty)), \tag{1.6}$$

and proved that all solutions of (1.5) oscillate if

$$\liminf_{t\to\infty}q(t)>\left(\frac{\sigma-\tau}{\sigma}\right)^{\sigma/\tau}\cdot\left(\frac{\sigma-\tau}{\tau}\right)^{-1}, \tag{1.7}$$

and Eq. (1.5) has a nonoscillatory solution if

$$\limsup_{t\to\infty}q(t)<\left(\frac{\sigma-\tau}{\sigma}\right)^{\sigma/\tau}\cdot\left(\frac{\sigma-\tau}{\tau}\right)^{-1}, \tag{1.8}$$

with the additional condition

$$|q(t') - q(t'')| \leq L|t' - t''| \ \text{ for any } \ t', t'' \in (0, \infty), \tag{1.9}$$

where $L > 0$ is some constant.

In the above mentioned papers the equations under consideration are called difference equations with continuous arguments (or continuous variables or continuous time) most likely because constant time delays appear in these equations.

In 1994, Golda and Werbowski [6] studied the second order linear functional equation of the form

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \quad t \geq 0, \tag{1.10}$$

where $P, Q : R^+ \to R^+, g : R^+ \to R \ (R^+ = [0, \infty))$ are given real valued functions, $g(t) \not\equiv t$ for $t \geq 0, \lim_{t\to\infty} g(t) = \infty$, and g^m denotes the m-th iterate of the function g, i.e.,

$$g^0(t) = t, \ \ g^{i+1}(t) = g(g^i(t)), \ \ t \geq 0, \ \ i = 0, 1, 2, \cdots,$$

and established several oscillation conditions. In particular, they proved that all solutions of Eq. (1.10) oscillate if

$$\liminf_{t\to\infty} Q(t)P(g(t)) > \frac{1}{4}. \tag{1.11}$$

It should be emphasized that condition (1.11) (*resp.* (1.7)) is a "sharp" condition in the sense that, when $P(t) \equiv p > 0, Q(t) \equiv q > 0$ and $g(t) = t - \tau, \tau > 0$ (*resp.* $q(t) \equiv q > 0$), it reduces to

$$pq > \frac{1}{4} \quad \left(resp. \ q > \left(\frac{\sigma - \tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma - \tau}{\tau}\right)^{-1} \right), \tag{1.12}$$

which is a necessary and sufficient condition for the oscillation of all solutions of

$$x(t - \tau) = px(t) + qx(t - 2\tau) \quad (resp. \ x(t) - x(t - \tau) + qx(t - \sigma) = 0)$$

because if we consider the last two equations then (1.4) reduces to the two conditions in (1.12) respectively.

Note that all the above mentioned papers deal with the oscillatory behavior only except [22] in which the nonoscillation conditions (1.8) and (1.9) were established for Eq. (1.5).

From the above discussion, the questions naturally arise as to whether the conditions

$$Q(t)P(g(t)) \leq 1/4 \ \text{ for large } t \tag{1.13}$$

and

$$q(t) \leq \left(\frac{\sigma - \tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma - \tau}{\tau}\right)^{-1} \ \text{ for large } t \tag{1.14}$$

imply that Eq. (1.10) and (1.5) have a nonoscillatory solution respectively.

The aim of this paper is to give answers to the above questions. We will prove that, under additional conditions on $g(t)$, condition (1.13) implies that Eq. (1.10) has a nonoscillatory solution. We will also prove that (1.14) is sufficient to guarantee the existence of a nonoscillatory solution of Eq. (1.5). It is to be noted that condition (1.9) is no longer required in our result and condition (1.14) is weaker than condition (1.8). The last result is given by considering the more general equation of the form

$$x(t) - px(t-\tau) + q(t)x(t-\sigma) = 0, \tag{1.15}$$

where $p \in (0, \infty)$ and τ, σ and $q(t)$ satisfy (1.6).

By a solution of (1.10) (*resp.* (1.15)) we understand a continuous real valued function $x : R^+ \to R$ such that $\sup\{|x(s)| : s \geq t_0\} > 0$ for any $t_0 \geq 0$ and x satisfies (1.10) (*resp.* (1.15)) on $[0, \infty)$. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Thus a nonoscillatory solution is either eventually positive or eventually negative.

2. MAIN RESULTS

2.1. Nonoscillation criteria for Eq. (1.10)

We will use the following hypotheses for Eq. (1.10).

(H_1) $P(t) \in C(R^+, (0, \infty)), Q(t) \in C(R^+, R^+)$;

(H_2) $g(t) \in C(R^+, R), g(0) = -r_1 \leq g(t) < t$ (retarded argument), $r_1 > 0, g(t) \to \infty$ as $t \to \infty$ and $g(t)$ is strictly increasing;

(H_3) $g(t) \in C(R^+, (0, \infty)), g(t) > t$ (advanced argument), and is strictly increasing.

Theorem 2.1. *Let (H_1) holds. Assume that either (H_2) or (H_3) is satisfied. If*

$$Q(t)P(g(t)) \leq 1/4 \quad for \ large \ t, \tag{1.13}$$

then Eq. (1.10) *has a nonoscillatory solution.*

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.1. *Consider the first order nonlinear functional equation*

$$u(t) = \frac{1}{1 - a(t)u(t-1)}, \quad t \geq 0, \tag{2.1}$$

where $a(t) \in C(R^+, R^+)$ is a given function. Assume that

$$a(t) \leq 1/4 \quad for \ large \ t. \tag{2.2}$$

Then Eq. (2.1) *has an eventually positive continuous solution $u(t)$.*

Proof. Without less of generality, we assume that

$$0 \leq a(t) \leq 1/4 \quad \text{for} \quad t \geq 0. \tag{2.3}$$

Set

$$\alpha = \begin{cases} \frac{1-\sqrt{1-4a(0)}}{2a(0)}, & \text{if } a(0) > 0 \\ 1, & \text{if } a(0) = 0. \end{cases} \tag{2.4}$$

Then α satisfies the relation

$$\alpha = \frac{1}{1 - a(0)\alpha}. \tag{2.5}$$

We claim that

$$1 \le \alpha \le 2. \tag{2.6}$$

Indeed, let

$$f(\xi) = \frac{1 - \sqrt{1-4\xi}}{2\xi}, \quad 0 < \xi \le 1/4.$$

Then

$$f'(\xi) = \frac{1 - 2\xi - \sqrt{1-4\xi}}{2\xi^2\sqrt{1-4\xi}}, \quad 0 < \xi < 1/4.$$

Set

$$F(\xi) = 1 - 2\xi - \sqrt{1-4\xi}, \quad 0 \le \xi \le 1/4.$$

Then

$$F'(\xi) = 2\left(\frac{1}{\sqrt{1-4\xi}} - 1\right) > 0, \quad 0 < \xi < 1/4.$$

Thus, $F(\xi)$ is strictly increasing on $(0, 1/4)$. Since $F(0) = 0$, it follows that $F(\xi) > 0$ for $0 < \xi \le 1/4$. Therefore, $f'(\xi) > 0$ for $0 < \xi < 1/4$. Noting that $f(1/4) = 2$ and

$$\lim_{\xi \to 0+} f(\xi) = \lim_{\xi \to 0} \frac{1 - \sqrt{1-4\xi}}{2\xi} = \lim_{\xi \to 0} \frac{1}{\sqrt{1-4\xi}} = 1,$$

we have $1 < f(\xi) \le 2$ for $0 < \xi \le 1/4$. This and (2.4) lead to (2.6).

Next, we define a function $u(t)$ as follows:

$$u(t) = \begin{cases} \frac{1}{1-a(0)\alpha}, & -1 \le t \le 0, \\ \frac{1}{1-a(t)u(t-1)}, & k < t \le k+1, \quad k = 0, 1, 2, ... \end{cases} \tag{2.7}$$

From (2.5), it is not difficult to see that

$$\lim_{t \to 0^+} u(t) = \frac{1}{1 - a(0)u(-1)} = \frac{1}{1 - a(0)\alpha} = u(0). \tag{2.8}$$

(2.7) and (2.8) imply that $u(t)$ is continuous on $[-1, \infty)$. We prove that

$$u(t) \ge 1 \quad \text{for} \quad t \ge -1. \tag{2.9}$$

Indeed, from (2.5), (2.6) and (2.7), we have

$$1 \le u(t) \le 2 \quad \text{for} \quad -1 \le t \le 0. \tag{2.10}$$

For $0 < t \le 1$, by (2.2), (2.7) and (2.9), we have

$$1 \le u(t) = \frac{1}{1 - a(t)u(t-1)} \le \frac{1}{1 - 2a(t)} \le 2.$$

In general we have $1 \le u(t) \le 2$ for $k < t \le k+1, k = 0, 1, 2, \ldots$. Thus, (2.9) holds. From (2.7) we see that

$$u(t) = \frac{1}{1 - a(t)u(t-1)}, \quad t \ge 0.$$

This shows that $u(t)$ is a positive continuous solution of (2.1). The proof is complete.

We now give some notations on the function $g(t)$. If $g(t)$ satisfies the condition (H_2), then $g^{-1}(t)$ $(g^{-1}(t) > t)$ denotes the inverse of the function $g(t)$ and $g^{-k}(t)$ is defined by $g^{-k-1}(t) = g^{-1}(g^{-k}(t)), k = 1, 2, \ldots$; If $g(t)$ satisfies the condition (H_3), then $g_{-1}(t)$ $(g_{-1}(t) < t)$ denotes the inverse of the function $g(t)$ and $g_{-k}(t)$ is defined by $g_{-k-1}(t) = g_{-1}(g_{-k}(t)), k = 1, 2, \ldots$.

Lemma 2.2. *Consider the first order nonlinear functional equation*

$$W(t) = \frac{1}{1 - b(t)W(g(t))}, \quad t \ge 0, \tag{2.11}$$

where $b(t) \in C(R^+, R^+)$ and $g(t)$ satisfies the condition (H_2). Then there exists a continuous change of variables that transforms Eq. (2.11) into Eq. (2.1). Such a change of variables is given by $u(t) = W(h(t)), t \ge 0$, and $a(t) = b(h(t))$, where $h(t)$ is defined by

$$h(t) = g^{-n}(\psi(t-n)), \quad n-1 \le t \le n, \quad n = 0, 1, 2, \ldots \tag{2.12}$$

and $\psi : [-1, 0] \to [-r_1, \infty)$ is any continuous increasing function satisfying the condition

$$g(\psi(0^-)) = \psi(-1^+). \tag{2.13}$$

Furthermore, we have that $u(\cdot)$ defined by $u(t) = W(h(t))$ oscillates if and only if $W(\cdot)$ oscillates.

Proof. Replacing t by $h(t)$ in (2.11) we have (cf.[1])

$$W(h(t)) = \frac{1}{1 - b(h(t))W(g(h(t)))}. \tag{2.14}$$

The term on the left side is just $u(t)$. To complete the transformation it suffices to have $a(t) = b(h(t))$ and $g(h(t)) = h(t-1)$, for $t \ge 0$. From (2.12), we have

$$h(t) = \psi(t), \quad -1 \le t \le 0,$$

$$h(t) = g^{-1}(h(t-1)), \quad n-1 \le t \le n, \quad n = 1, 2, \ldots$$

By (2.13) we see that h is continuous. Since ψ is increasing on $[-1, 0]$, it follows that h is increasing. Finally, to see that $u(\cdot)$ oscillates if and only if $W(\cdot)$ oscillates, it suffices to prove that $h(t) \to \infty$ as $t \to \infty$. Indeed, if $u(\cdot)$ oscillates, then there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $u(t_n) = 0$. Let $s_n = h(t_n)$, then $s_n \to \infty$ as $n \to \infty$ because $h(t) \to \infty$ as $t \to \infty$. Thus, $W(s_n) = W(h(t_n)) = u(t_n) = 0$. This shows that $W(\cdot)$ oscillates. Conversely, if $W(\cdot)$ oscillates, then there exists a sequence $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$ and $W(s_n) = 0$. Let $t_n = h^{-1}(s_n)$ (here h^{-1} is the inverse of the function h), then $t_n \to \infty$ as $n \to \infty$ and $u(t_n) = W(h(t_n)) = W(s_n) = 0$. Thus, $u(\cdot)$ oscillates. Now, to prove that $h(t) \to \infty$ as $t \to \infty$, we need only to prove that $h(n) \to \infty$ as $n \to \infty$, where n takes only integer values. Otherwise, the sequence $h(n) = g^{-n}(\psi(0))$ has a limit L, then

$$g^{-1}(L) = g^{-1}\left(\lim_{n\to\infty} g^{-n}(\psi(0))\right) = \lim_{n\to\infty} g^{-(n+1)}(\psi(0)) = L.$$

This is impossible because $g^{-1}(t) > t$ for all t. The proof is complete.

Remark 2.1. One way to transform (2.11) into (2.1) is to suppose that the function ψ has the form $\psi(t) = at + b$, where a and b are to be determined. We first require $\psi(-1) = -r_1$, which gives $b - a = -r_1$. In addition, condition (2.13) requires $g(b) = -a + b$. From (H_2), $g(0) = -r_1$, it follows that $b = 0, a = r_1$. Thus $\psi(t) = r_1 t$.

Lemma 2.3. *Assume that* (H_1) *and* (H_2) *hold. Then Eq.* (1.10) *has an eventually positive solution if and only if the first order nonlinear functional equation*

$$W(t) = \frac{1}{1 - Q(t)P(g(t))W(g(t))}, \quad t \geq 0 \tag{2.15}$$

has an eventually positive continuous solution.

Proof. Assume that $x(t)$ is an eventually positive solution of Eq. (1.10). Dividing both sides of (1.10) by $x(g(t))$ gives

$$1 = P(t)\frac{x(t)}{x(g(t))} + Q(t)\frac{x(g^2(t))}{x(g(t))}. \tag{2.16}$$

Set

$$\overline{W}(t) = \frac{x(g(t))}{P(t)x(t)}. \tag{2.17}$$

Then $\overline{W}(t)$ is eventually positive and continuous and satisfies

$$1 = \frac{1}{\overline{W}(t)} + Q(t)P(g(t))\overline{W}(g(t)), \tag{2.18}$$

which shows that $\overline{W}(t)$ is an eventually positive continuous solution of (2.15).

Next assume that $W(t)$ is an eventually positive continuous solution of (2.15). Without loss of generality, we may assume that $W(t) > 0$ for $t \geq 0$. By similar arguments, as in the proof of Lemma 2.2, we see that there exists a continuous change of variables that transforms the equation

$$x(t) = \frac{1}{W(t)P(t)} x(g(t)), \quad t \geq 0 \tag{2.19}$$

into the equation

$$y(t) = R(t)y(t-1), \quad t \geq 0, \tag{2.20}$$

where $R(t) = [W(h(t))P(h(t))]^{-1}, h(t)$ is as in Lemma 2.2, and Eq. (2.19) has an eventually positive continuous solution $x(t)$ if and only if Eq. (2.20) has an eventually positive continuous solution $y(t)$. Since $R(t) > 0$ for $t \geq 0$, it is easy to see that Eq. (2.20) has an eventually positive continuous solution. Indeed, the function $y(t)$ defined by

$$y(t) = \begin{cases} r(t), & -1 \leq t \leq 0, \\ R(t)y(t-1), & k < t \leq k+1, \quad k = 0, 1, 2, ..., \end{cases}$$

where $r(t)$ is any positive continuous function on $[-1, 0]$ such that $r(0) = R(0)r(-1)$, is a positive continuous solution of (2.20). Thus, Eq. (2.19) has an eventually positive continuous solution. Let $\overline{x}(t)$ be such a solution. Substituting

$$W(t) = \frac{\overline{x}(g(t))}{\overline{x}(t)P(t)}$$

into (2.15), we obtain

$$\frac{\overline{x}(g(t))}{\overline{x}(t)P(t)} \left(1 - Q(t)P(g(t)) \frac{\overline{x}(g^2(t))}{\overline{x}(g(t))P(g(t))}\right) = 1,$$

i.e.,

$$\overline{x}(g(t)) = P(t)\overline{x}(t) + Q(t)\overline{x}(g^2(t)).$$

This shows that $\overline{x}(t)$ is a positive solution of (1.10). The proof is complete.

Proof of Theorem 2.1. We first consider the case when $g(t)$ satisfies (H_2). By Lemma 2.3, it suffices to prove that Eq. (2.15) has an eventually positive continuous solution. Set $b(t) = Q(t)P(g(t))$ and let $h(t)$ be a change of variables as in Lemma 2.2. Define $a(t) = b(h(t)) = Q(h(t))P(g(h(t)))$ and $u(t) = W(h(t))$. From condition (1.13), we have

$$0 \leq a(t) = Q(h(t))P(g(h(t))) \leq 1/4 \quad \text{for large} \quad t.$$

Thus, by Lemma 2.1, Eq. (2.1) has an eventually positive continuous solution $u(t)$. By Lemma 2.2, we see that Eq. (2.15) has an eventually positive continuous solution $W(t)$.

Next, we consider the case when $g(t)$ satisfies (H_3). Since $g(t)$ satisfies (H_3), it follows that $g_{-1}(t)$ satisfies (H_2), with the possible exception that $g_{-1}(0) = -r_2 < 0, r_2 \neq r_1$. Replacing $g^2(t)$ by t in Eq. (1.10), we have

$$x(g_{-1}(t)) = Q(g_{-2}(t))x(t) + P(g_{-2}(t))x(g_{-2}(t)). \tag{2.21}$$

Condition (1.13) implies that for t sufficiently large

$$0 \leq P(g_{-2}(t))Q(g_{-2}(g_{-1}(t))) = Q(g_{-3}(t))P(g_{-2}(t)) \leq 1/4.$$

As in the case when $g(t)$ satisfies (H_2), we see that Eq. (2.21) has an eventually positive solution. Thus, Eq. (1.10) has an eventually positive solution. The proof is complete.

Example 2.1. Consider the equation

$$x(t/2) = e^t x(t) + \frac{1}{4}e^{-t/2}x(t/4). \tag{2.22}$$

It is easy to see that

$$Q(t)P(g(t)) = \frac{1}{4}e^{-t/2} \cdot e^{t/2} = \frac{1}{4}.$$

Thus, by Theorem 2.1, Eq. (2.22) has a nonoscillatory solution. In fact, $x(t) = t^{-1}e^{-2t}$ is such a nonoscillatory solution.

2.2. Nonoscillation criteria for Eq. (1.15)

We will establish the following nonoscillation theorem for Eq. (1.15).

Theorem 2.2. *Let $\sigma > \tau$. Assume that*

$$p^{-\sigma/\tau} \cdot q(t) \leq \left(\frac{\sigma - \tau}{\sigma}\right)^{\sigma/\tau} \cdot \left(\frac{\sigma - \tau}{\tau}\right)^{-1} \quad \textit{for large } t. \tag{2.23}$$

Then Eq. (1.15) *has a nonoscillatory solution.*

Remark 2.2. When $p = 1$, condition (2.23) reduces to condition (1.14) and Eq. (1.15) reduces to Eq. (1.5). Thus condition (1.14) is sufficient for Eq. (1.5) to have a nonoscillatory solution. On the other hand, condition (2.23) is "sharp" in the sense that when $q(t) \equiv q > 0$ condition (2.23) also is necessary for Eq. (1.15) to have a nonoscillatory solution (cf.(1.4)).

To prove this theorem, we need two intermediate results. The first one is Schauder's fixed-point theorem [16].

Lemma 2.4. *Let Ω be a nonempty bounded closed convex subset of a Banach space $(B, ||\cdot||)$, and let $S : \Omega \to \Omega$ be a continuous mapping such that $S(\Omega)$ is (relatively) compact (S is completely continuous). Then, $S(x) = x$, for some $x \in \Omega$.*

The second one is the following version of Ascoli's theorem.

Lemma 2.5. *Let $\{f_n : [0, \infty) \to R, n = 1, 2, ...\}$ be a sequence of functions such that:*

(i). *there exists a constant $M > 0$ such that $|f_n(t)| \leq M$ for all $n \geq 1$ and $t \geq 0$;*

(ii). *$f_n(t)$ is continuous on $[0, \infty)$ for all $n \geq 1$;*

(iii). *there exist constants $L > 0, \mu > 0$ such that $0 \leq f_n(t) \leq Le^{-\mu t}$ for $t \geq T > 0$ and all $n \geq 1$, where T is a constant.*

Then there exists a continuous function $f : [0, \infty) \to R$ and a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n(t) \to f(t)$ as $n \to \infty$, uniformly on $[0, \infty)$.

Remark 2.3. The authors are unaware of a precise reference for Lemma 2.5, but it is a sequence of the results in [8].

Proof of Theorem 2.2. Let the right side of (2.23) be c. Then, by the results in [12] (see also condition (1.4)), the equation

$$u(t) - u(t-\tau) + cu(t-\sigma) = 0 \tag{2.24}$$

has an nonoscillatory solution of the form $u(t) = e^{\lambda t}$, where $\lambda < 0$ is a root of the characteristic equation $1 - e^{-\lambda\tau} + ce^{-\lambda\sigma} = 0$. It is clear that the equality holds:

$$u(t) = c\sum_{i=1}^{\infty} u(t + i\tau - \sigma).$$

For each real number r, let us define B_r as the space of all real bounded continuous functions defined on $[r, \infty)$, provided with the usual sup-norm; and let $\Omega_r := \{v \in B_r : 0 \le v(t) \le e^{\lambda t}, t \ge r\}$. It is clear that Ω_r is a nonempty bounded closed convex subset of the Banach space B_r. Let $T > 0$ be such that (2.23) holds for $t \ge T$. Define a mapping S on Ω_T as follows:

$$S(y)(t) = \begin{cases} \sum_{i=1}^{\infty} q^*(t+i\tau)y(t+i\tau-\sigma), & t \ge T+\sigma-\tau, \\ S(y)(T+\sigma-\tau) + u(t) - u(T+\sigma-\tau), & T \le t < T+\sigma-\tau, \end{cases}$$

where

$$q^*(t) = p^{-\sigma/\tau} \cdot q(t) \le c, \quad t \ge T.$$

Then for $t \ge T + \sigma - \tau$

$$0 \le S(y)(t) \le \sum_{i=1}^{\infty} cu(t + i\tau - \sigma) = u(t) = e^{\lambda t}. \tag{2.25}$$

While for $T \le t < T + \sigma - \tau$, we also have

$$0 \le S(y)(t) \le u(T+\sigma-\tau) + u(t) - u(T+\sigma-\tau) = u(t) \le e^{\lambda t}.$$

Thus, (2.25) holds for $t \ge T$. For any $y \in \Omega_T$, we claim that $S(y)$ is continuous. Since $\lim_{t\to\infty} u(t) = 0$, it follows that for any $\varepsilon > 0$ there exists $T_1 > T$ such that $u(t) < \varepsilon$ for $t > T_1$. Choose a positive integer N such that $N\tau \ge T_1$. Then for all $t \ge T + \sigma - \tau$ we have

$$\begin{aligned} \sum_{i=m+1}^{n} q^*(t+i\tau)y(t+i\tau-\sigma) &\le \sum_{i=m+1}^{\infty} cu(t+i\tau-\sigma) \\ &= u(t+m\tau) < \varepsilon \end{aligned}$$

for any $m, n \ge N$, which implies that the series $\sum_{i=1}^{\infty} q^*(t+i\tau)y(t+i\tau-\sigma)$ converges uniformly on $[T + \sigma - \tau, \infty)$. Thus, $S(y)$ is continuous. From this and (2.25), we have $S(\Omega_T) \subset \Omega_T$.

Notice that $0 \le S(y)(t) \le e^{\lambda t}$. This and Lemma 2.5 imply that $S(\Omega_T)$ is (relatively) compact. Hence, by Lemma 2.4, $S(y) = y$ for some $y \in \Omega_T$. i.e.,

$$y(t) = \begin{cases} \sum_{i=1}^{\infty} q^*(t+i\tau)y(t+i\tau-\sigma), & t \ge T+\sigma-\tau, \\ y(T+\sigma-\tau)+u(t)-u(T+\sigma-\tau), & T \le t < T+\sigma-\tau. \end{cases} \tag{2.26}$$

and

$$y(t) - y(t-\tau) + q^*(t)y(t-\sigma) = 0, \quad t \ge T+\sigma. \tag{2.27}$$

We claim that $y(t) > 0$ for $t \ge T$. Since $u'(t) < 0, t \ge T$, from (2.26), we have

$$y(t) > 0, \quad T \le t < T+\sigma-\tau.$$

Assume that there exists a $t \in [T+\sigma-\tau, \infty)$ such that $y(t) \le 0$, then we can let

$$t^* = \inf\{t \ge T+\sigma-\tau : y(t) \le 0\},$$

so that

$$y(t^*) = 0 \quad \text{and} \quad y(t) > 0, \quad T \le t < t^*.$$

On the other hand, from (2.26), we have

$$\begin{aligned} y(t^*) &= \sum_{i=1}^{\infty} q^*(t^*+i\tau)y(t^*+i\tau-\sigma) \\ &\ge q^*(t^*+\tau)y(t^*+\tau-\sigma) > 0, \end{aligned}$$

a contradiction. Thus $y(t) > 0$ for $t \ge T$. Finally, let us define $x(t) = p^{t/\tau}y(t)$. Then, by (2.27), we have

$$x(t) - px(t-\tau) + q(t)x(t-\sigma) = 0.$$

Thus, $x(t)$ is a nonoscillatory solution of (1.15). The proof is complete.

Example 2.2. Consider the equation

$$x(t) - x(t-1) + \frac{(et-t+1)(2t-11)}{2t(t-1)e^{5.5}}x(t-5.5) = 0.$$

It is not difficult to check that for $t \ge 6$

$$\begin{aligned} q(t) &= \frac{(et-t+1)(2t-11)}{2t(t-1)e^{5.5}} \\ &\le \left(\frac{5.5-1}{5.5}\right)^{5.5} \cdot (5.5-1)^{-1}. \end{aligned}$$

Thus, by Theorem 2.2, this equation has a nonoscillatory solution. In fact, $x(t) = t^{-1}e^{-t}$ is such a nonoscillatory solution.

REFERENCES

1. T.H. Barr, Oscillations in linear delay differential equations, *J. Math. Anal. Appl.*, **195** (1995), 261-277.

2. Y. Domshlak, Oscillatory properties of linear difference equations with continuous time, *Differential Equations and Dynamical Systems*, **4** (1993), 311-324.

3. *Á.* Elbert and I.P. Stavroulakis, Oscillations of first order differential equations with deviating arguments, University of Ioannina, T. R. N^0 172 1990, *Recent treads in differential equations, 163-178, World Sci. Ser. Appl. Anal., 1, World Sci. Publishing Co.* (1992).

4. L.H. Erbe and B.G. Zhang, Oscillation of discrete analogues of delay equations, *Diff. and Integral Equations*, **2** (1989), 300-309.

5. N. Fukagai and T. Kusano, Oscillation theory of first order functional differential equations with deviating arguments, *Ann. Mat. Pura Appl.*, **136** (1984), 95-117.

6. W. Golda and J. Werbowski, Oscillation of linear functional equations of the second order, *Funkcial. Ekvac.*, **37** (1994), 221-227.

7. I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

8. H.P. Heinz, Theorems of Ascoli type involving measures of noncompactness, *Nonlinear Anal.*, **5** (1981), 277-286.

9. R.G. Koplatadze and T.A. Chanturija, On oscillatory and monotonic solutions of first order differential equations with deviating arguments, *Differential'nye Uravnenija*, **18** (1982), 1463-1465 (Russian).

10. E. Kozakiewicz, Conditions for the absence of positive solutions of a first order differential inequality with a single delay, *Archivum Mathematicum*, **31** (1995), 291-297.

11. G. Ladas, Recent development in the oscillation of delay difference equations, *International Conference of Differential Equations, Colorado Springs*, USA. June, 1989.

12. G. Ladas, L. Pakula and Z. Wang, Necessary and sufficient conditions for the oscillation of difference equations, *PanAmerican Math. J.*, **2** (1992), 17-26.

13. G.S. Ladde, V. Lakshmikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York, 1987.

14. B.S. Lalli and B.G. Zhang, Oscillation of difference equations, *Colloquium Mathematicum*, **LXV** (1993), 25-32.

15. W. Nowakowska and J. Werbowski, Oscillation of linear functional equations of higher order, *Archivum Math.*, **31** (1995), 251-258.

16. J. Schauder, Der Fixpunktsatz in Funktionalraumen, *Studia Mathematica*, **2** (1930), 171-180.

17. J.H. Shen, Comparison theorems for the oscillation of difference equations with continuous arguments and applications, *Chinese Science Bulletin*, **41** (1996), 1506-1510.

18. I.P. Stavroulakis, Oscillation of delay difference equations, *Computers Math. Applic.*, **29** (1995), 83-88.

19. J. Yan and F. Zhang, Oscillation for system of delay difference equations, *J. Math. Anal. Appl.*, **230** (1999), 223-231.

20. Y. Zhang and J. Yan, Oscillation criteria for difference equations with continuous arguments, *Acta Math. Sinica.*, **38** (1995), 406-411 (in Chinese).

21. B.G. Zhang, Jun Yan and S.K. Choi, Oscillation for difference equations with continuous variable, *Computers Math. Applic.*, **36** (1998), 11-18.

22. Y. Zhang, J. Yan and A. Zhao, Oscillation criteria for a difference equation, *Indian J. Pure Appl. Math.*, **28** (1997), 1241-1249.